JOURNAL OF APPROXIMATION THEORY **63**, 375–383 (1990)

Best Piecewise Monotone Uniform Approximation

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Communicated by Frank Deutsch

Received February 13, 1989; revised April 16, 1990

The problem considered is that of finding a best uniform approximation to a real function $f \in C[a, b]$ from the class of piecewise monotone functions. The existence, characterization, and nonuniqueness of best approximations are established.

1. INTRODUCTION

Let $I = [a, b]$ be a compact real interval and $B = B(I)$ (resp. $C = C(I)$) be the Banach space of all bounded (resp. continuous) real functions f on I with the uniform norm $\|f\| = \sup\{|f(x)| : x \in I\}$. For any integer $n \geqslant 1$, let

$$\Omega = \{p = (p_0, p_1, ..., p_n) \in R^{n+1} : a = p_0 \leqslant p_1 \leqslant \cdots \leqslant p_n = b\}.$$

Then, Ω is compact in R^{n+1}. Given a $p \in \Omega$, define intervals $I_j = [p_{j-1}, p_j)$, for $1 \leqslant j \leqslant n-1$, and $I_n = [p_{n-1}, p_n]$. Let

$$K(p) = \{h \in B: (-1)^j h \text{ is nonincreasing on } I_j,\ 1 \leqslant j \leqslant n\}.$$

$K(p)$ is called the set of all n-piecewise monotone functions with the knot vector p. Some functions in $K(p)$ have more than one knot vector. In general, the set of all knot vectors for a given function in $K(p)$ is a convex subset of Ω, as may be easily seen. Next, let $K = \bigcup \{K(p) : p \in \Omega\}$. We call

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0021-9045/90 $3.00

K the set of n-piecewise monotone functions, or simply, the set of piecewise monotone functions. We point out that several entities defined in this article including $K(p)$ and K depend on n although such dependence is suppressed in their notation. Given f in C, a function g in $E \subset B$ is called a best approximation to f from E if $\|f-g\| = \inf\{\|f-h\| : h \in E\}$. In this article, we consider the problem of finding a best approximation to f in C from $K(p)$, K, $K(p) \cap C$, and $K \cap C$.

In Section 2, we analyze some properties of $K(p)$ and K, and establish the existence and characterization of a best approximation to f in C from these sets. In Section 3, we demonstrate the existence and nonuniqueness of best approximations from $K(p) \cap C$ and $K \cap C$. The latter obviously implies the nonuniqueness of best approximations from $K(p)$ and K.

If $n = 1$, the above problem is included in the best uniform monotone approximation investigated in [8]. If $n = 2$, the problem is a slightly restricted version of the best uniform quasi-convex approximation studied in [10, 11, 13]. Quasi-convex functions are considered in [2, 3, 5]. A slightly general concept of piecewise monotone functions is used in [12] to establish the existence of a best L_p approximation from various function classes including quasi-convex, convex, n-convex, and star-shaped. Additional references on piecewise monotone functions are [1, 6, 7].

2. Best Approximation from $K(p)$ and K

In this section we establish some properties of $K(p)$ and K, and also the existence and characterization of best approximations from these sets.

Let $f \in B$. For $E \subset B$, let $\rho(f, E) = \inf\{\|f-h\| : h \in E\}$. Define $\Delta(p) = \rho(f, K(p))$, $p \in \Omega$, and $\Delta^* = \rho(f, K)$. Then, $\Delta^* = \inf\{\Delta(p) : p \in \Omega\}$. We denote by $K^*(p)$ (resp. K^*) the set of all best approximations to f from $K(p)$ (resp. K). Let also $P^* = \{p \in \Omega : \Delta(p) = \Delta^*\}$. We call P^* the set of best knot vectors for piecewise monotone approximation to f; this terminology is justified by Theorem 2.3(b) below. The left-hand and right-hand limits of h at x are denoted by $h(x^-)$ and $h(x^+)$, respectively.

Theorem 2.1 (Existence of a best approximation from K and a best knot vector). *If $f \in C$, then there exist $p^* \in P^*$ and $g^* \in K(p^*) \cap K^*$. Thus $P^* \neq \varnothing$.*

Proof (As in [9] or [12]). Let $T(h)$ denote the total variation of a function h on I. If $h \in K$, then there exists $q \in \Omega$ such that $h \in K(q)$. We have,

$$T(h) = \sum_{i=1}^{n} |h(q_i^-) - h(q_{i-1})| + \sum_{i=1}^{n} |h(q_i^-) - h(q_i)| \leqslant 4n\|h\|.$$

Now for each positive integer k, there exists $h_k \in K$ such that $\|f-h_k\| \leqslant \Delta^* + 1/k$. Hence, $\|h_k\| \leqslant \|f\| + \Delta^* + 1$. Then $h_k \in K(q^{(k)})$ and, by the above result, $T(h_k) \leqslant 4n\,\|h_k\| \leqslant 4n(\|f\| + \Delta^* + 1)$. By the compactness of Ω and Helly's selection theorem [4, p. 222], there exist subsequences $p^{(k)}$ and g_k of $q^{(k)}$ and h_k, respectively, so that $p^{(k)} \to p^*$ and $g_k \to g^*$ pointwise. Then $\|f-g^*\| \leqslant \Delta^*$. Clearly, g^* may not be in $K(p^*)$. We redefine g^* at the points p_i^*, $1 \leqslant i \leqslant n-1$, to equal its right-hand limits at these points. Then, $g^* \in K(p^*)$ and, by the continuity of f we have $\|f-g^*\| = \Delta^*$. Thus $g^* \in K^*$. Also, $\Delta^* = \Delta(p^*)$ and $p^* \in P^*$. The proof is complete.

We now establish some properties of $K(p)$ and K. A subset E of B is called a cone if $f \in E$ implies that $\lambda f \in E$ for all $\lambda \geqslant 0$. It is easy to show that a cone E is convex if and only if $f + h \in E$ whenever $f, h \in E$. Note that if $n = 1$, then $\Omega = \{p\}$ and $K = K(p)$, where $p = (a, b)$. In the next proposition, the topology under consideration is the one generated by the uniform norm.

PROPOSITION 2.1. (a) *For all* $p \in \Omega$ *and* $n \geqslant 1$, $K(p)$ *and* $K(p) \cap C$ *are closed convex cones.*

(b) *For* $n = 1$, K *and* $K \cap C$ *are closed convex cones. For* $n \geqslant 2$, K *and* $K \cap C$ *are cones which are not convex,* K *is not closed, but* $K \cap C$ *is closed.*

Proof. We first show that $K \cap C$ is closed. Let $h_k \in K \cap C$ and $h \in B$ with $\|h-h_k\| \to 0$ as $k \to \infty$. Then, by uniform convergence, $h \in C$. As in Theorem 2.1, there exists a subsequence g_k of h_k with $g_k \in K(p^{(k)})$ so that $g_k \to g^*$ pointwise and $p^{(k)} \to p^*$. Then $h = g^*$ and, by continuity, $h \in K(p^*)$. Hence $h \in K \cap C$ and $K \cap C$ is closed.

We now show that K is not closed for $n \geqslant 2$; it suffices to do so for $n = 2$. Let $n = 2$, $I = [-1, 1]$, $f(x) = 0$ for $-1 \leqslant x \leqslant 0$, and $= 1 - x$ for $0 < x \leqslant 1$. Clearly, $f \in B \backslash K$. Let $q^{(k)} = (-1, 1/k, 1)$, $k \geqslant 2$. Define $f_k(x) = 0$ for $-1 \leqslant x \leqslant 0$, $= 1$ for $0 < x < 1/k$, and $= 1 - x$ for $1/k \leqslant x \leqslant 1$. Then $f_k \in K(q^{(k)}) \subset K$, and $\|f - f_k\| = 1/k \to 0$ as $k \to \infty$. Hence f is in the closure of K. Thus K is not closed.

The remaining assertions may be established directly from the definitions by elementary methods. The proof is complete.

We note that the cone of quasi-convex functions considered in [10, 11, 13] is closed and contains the cone K for $n = 2$, and the latter is not closed as shown above.

Let $f \in B$. If $a \leqslant x \leqslant y \leqslant b$ and $1 \leqslant k \leqslant n$, define

$$F_k(x, y) = (f(x) - f(y))/2, \qquad k \text{ odd},$$
$$= (f(y) - f(x))/2, \qquad k \text{ even}.$$

For any interval $J \subset I$, let $d_k(J) = \sup\{F_k(x, y): x, y \in J, x \leqslant y\}$. Note that, when k is odd (resp. even), $d_k(J)$ is a measure of the extent by which f fails to be nondecreasing (resp. nonincreasing) on J. For $p \in \Omega$, let $\delta(p) = \max\{d_k(I_k): 1 \leqslant k \leqslant n\}$. Let also $\delta^* = \inf\{\delta(p): p \in \Omega\}$ and $P = \{p \in \Omega: \delta(p) = \delta^*\}$, the subset of Ω on which δ attains its minimum.

PROPOSITION 2.2. *Let $f \in B$.*

(a) *$\delta(p) = 0$ for some $p \in \Omega$ if and only if $f \in K(p)$.*

(b) *If $f \in K$ then $\delta^* = 0$.*

(c) *If $f \in C$ then $\delta(p)$ is a continuous function of $p \in \Omega$. Hence, $\delta^* = 0$ if and only if $f \in K \cap C$.*

Proof. (a) This follows directly from the definition of $K(p)$.

(b) If $f \in K$ then $f \in K(p)$ for some $P \in \Omega$ and $\delta(p) = 0$. Hence $\delta^* = 0$.

(c) For $p \in \Omega$, let $\|p\|$ denote the Euclidean norm. Let $\varepsilon > 0$. Since $f \in C$, there exists $\rho > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \rho$. Let $p, q \in \Omega$ with $\|p - q\| < \rho$. Then $|p_k - q_k| < \rho$ for all $0 \leqslant k \leqslant n$. Hence, if $I_k = [p_{k-1}, p_k)$ and $J_k = [q_{k-1}, q_k)$, then we have $|d_k(I_k) - d_k(J_k)| \leqslant \varepsilon$, as may be easily shown. Hence, $|\delta(p) - \delta(q)| \leqslant \varepsilon$ and δ is continuous. Now, if $\delta^* = 0$ then there exists p with $\delta^* = \delta(p) = 0$. Then $f \in K(p) \subset K$. The proof is complete.

The example of Proposition 2.1 also illustrates that, in general, the continuity condition of f in Theorem 2.1 cannot be dropped and the converse of Proposition 2.2(b) is not true. For f defined in the proof of Proposition 2.1, $\Delta^* = 0$ and a best approximation from K does not exist. Also, $\delta(q^{(k)}) \to 0$ as $k \to \infty$. Hence $\delta^* = 0$ but $f \in B \backslash K$.

For $p \in \Omega_n$ and $1 \leqslant j \leqslant n$, define $\underline{g}_p$, $\bar{g}_p$ in K by

$$\begin{aligned} \underline{g}_p(x) &= \sup\{f(y): p_{j-1} \leqslant y \leqslant x\} - \delta(p), x \in I_j, \qquad j \text{ odd} \\ &= \sup\{f(y): x \leqslant y \leqslant p_j\} - \delta(p), x \in I_j, \qquad j \text{ even,} \end{aligned}$$

and

$$\begin{aligned} \bar{g}_p(x) &= \inf\{f(y): x \leqslant y \leqslant p_j\} + \delta(p), x \in I_j, \qquad j \text{ odd,} \\ &= \inf\{f(y): p_{j-1} \leqslant y \leqslant x\} + \delta(p), x \in I_j, \qquad j \text{ even.} \end{aligned}$$

THEOREM 2.2 (Best Approximation from $K(p)$). *Let $p \in \Omega$, $f \in B$ and $g \in K(p)$. Then the following holds.*

(a) (*Duality*) *$\Delta(p) = \delta(p)$.*

(b) (*Existence and characterization*) *Both $\underline{g}_p$, $\bar{g}_p \in K^*(p)$ with $\underline{g}_p \leqslant \bar{g}_p$. Furthermore, $g \in K^*(p)$ if and only if $\underline{g}_p \leqslant g \leqslant \bar{g}_p$.*

Proof. Applying [8, part I, Theorem 2.1] with w identically equal to 1 to each interval I_j we obtain the required results. The arguments given there hold even if some of the intervals are half-open. The maximum of θ's for all intervals gives $\delta(p)$. The proof is complete.

THEOREM 2.3 (Best Approximation from K). *Let $f \in C$ and $g \in K$. Then the following holds.*

(a) *(Duality)* $\Delta^* = \delta^*$.

(b) *(Optimal knots)* $P^* = P = \{p \in \Omega: K(p) \cap K^* \neq \varnothing\} \neq \varnothing$.

(c) *(Characterization)* $g \in K^*$ *if and only if there exists a* $p \in P$ *such that* $\underline{g}_p \leqslant g \leqslant \bar{g}_p$. *(Both* $\underline{g}_p$ *and* $\bar{g}_p$ *are in* K^*.*) Consequently,* $K^* = \bigcup \{[\underline{g}_p, \bar{g}_p]: p \in P\}$, *where* $[\underline{g}, \bar{g}_p]$ *denotes the "function interval"* $\{g \in K: \underline{g}_p \leqslant g \leqslant \bar{g}_p\}$.

Proof. (a) and (b) By Theorem 2.2(a), for each $p \in \Omega$, $\Delta(p) = \delta(p)$. Hence, $\Delta^* = \delta^*$ and $P^* = P$. There latter is nonempty by Theorem 2.1. Now, by the definition of P^*, we have $P^* \supset \{p \in \Omega: K(p) \cap K^* \neq \varnothing\}$. If $p \in P^*$, then, by Theorem 2.2(b), there exists a best approximation g to f from $K(p)$ with $\|f - g\| = \Delta(p) = \Delta^*$. Hence, $g \in K^*$ and (b) is established.

(c) Note that $g \in K^*$ if and only if $g \in K^*(p)$, where $p \in P^* = P$. The result now follows from Theorem 2.2(b).

The proof is complete.

3. BEST APPROXIMATION FROM $K(p) \cap C$ AND $K \cap C$

In this section we obtain the existence, characterization, and nonuniqueness of best approximations from $K(p) \cap C$ and $K \cap C$.

We first define some notation. Let $f \in C$. For $0 \leqslant x \leqslant y \leqslant 1$, let $m(x, y) = \min\{f(z): x \leqslant z \leqslant y\}$ and $M(x, y) = \max\{f(z): x \leqslant z \leqslant y\}$. By the continuity of f, for any (open, half-open, or closed) nonempty subinterval J of I with endpoints x, y, we have $d_k(J) = d_k([x, y])$, for all $1 \leqslant k \leqslant n$. For convenience, in the rest of the exposition, we denote $d_k([x, y])$ by $d_k(x, y)$. We now establish two basic results.

LEMMA 3.1. *Let $f \in C$, $p \in P$, and k be a fixed integer with $1 \leqslant k \leqslant n - 1$.*

(a) *Let* $p_k^{(1)} \in [p_{k-1}, p_{k+1}]$ *such that* $f(p_k^{(1)}) = M(p_{k-1}, p_{k+1})$ *for odd* k, *and* $f(p_k^{(1)}) = m(p_{k-1}, p_{k+1})$ *for even* k. *Then,* $d_k(p_{k-1}, p_k^{(1)}) \leqslant \delta^*$ *and* $d_{k+1}(p_k^{(1)}, p_{k+1}) \leqslant \delta^*$.

(b) *Let* $p_k^{(1)} \in [p_k, p_{k+1}]$ *such that the following* (i) *and* (ii) *hold.*

(i) *for odd* k, $f(p_k^{(1)}) = M(p_{k-1}, p_{k+1})$, $m(p_k, p_k^{(1)}) < f(p_{k-1}) = m(p_{k-2}, p_k)$, *and* $p_{k-1}^{(1)} = \inf\{z \in [p_k, p_k^{(1)}]: f(z) = m(p_k, p_k^{(1)})\}$,

(ii) *for even* k, $f(p_k^{(1)}) = m(p_{k-1}, p_{k+1})$, $m(p_k, p_k^{(1)}) < f(p_{k-1}) = M(p_{k-2}, p_k)$, *and* $p_{k-1}^{(1)} = \inf\{z \in [p_k, p_k^{(1)}]: f(z) = M(p_k, p_k^{(1)})\}$.

Then, $d_k(p_{k-1}^{(1)}, p_k^{(1)}) \leqslant \delta^*$ *and* $d_{k-1}(p_{k-2}, p_{k-1}^{(1)}) \leqslant \delta^*$.

Proof. (a) We present only the proof for odd k. If $p_k^{(1)} = p_k$, then the result holds by the definition of d_k. If $p_k^{(1)} < p_k$, then $d_k(p_{k-1}, p_k^{(1)}) \leqslant d_k(p_{k-1}, p_k) \leqslant \delta^*$. Assume $d_{k+1}(p_k^{(1)}, p_{k+1}) > \delta^*$. Then, there exist two points $x < y$ in $[p_k^{(1)}, p_{k+1}]$ such that $f(y) - f(x) > 2\delta^*$. If $p_k^{(1)} \leqslant x \leqslant p_k$, then we have,

$$d_k(p_{k-1}, p_k) \geqslant (f(p_k^{(1)}) - f(x))/2 \geqslant (f(y) - f(x))/2 > \delta^*,$$

which is a contradiction. Similarly, if $p_k < x \leqslant p_{k+1}$, then $d_{k+1}(p_k, p_{k+1}) \geqslant (f(y) - f(x))/2 > \delta^*$, a contradiction. The case $p_k^{(1)} > p_k$ can be handled similarly to obtain a contradiction.

(b) We prove this result only for odd k. Since $p_{k-1}^{(1)} \in [p_k, p_k^{(1)}]$, by (a),

$$d_k(p_{k-1}^{(1)}, p_k^{(1)}) \leqslant d_k(p_{k-1}, p_k^{(1)}) \leqslant \delta^*.$$

Also,

$$\begin{aligned} &d_{k-1}(p_{k-2}, p_{k-1}^{(1)}) \\ &\quad \leqslant \max\{d_{k-1}(p_{k-2}, p_{k-1}), d_{k-1}(p_{k-1}, p_{k-1}^{(1)}), \\ &\qquad \sup\{(f(y) - f(x))/2: p_{k-2} \leqslant x \leqslant p_{k-1}, p_{k-1} \leqslant y \leqslant p_{k-1}^{(1)}\}\}. \end{aligned}$$

But

$$\begin{aligned} d_{k-1}(p_{k-1}, p_{k-1}^{(1)}) &\leqslant (M(p_{k-1}, p_{k-1}^{(1)}) - f(p_{k-1}^{(1)}))/2 \\ &\leqslant (f(p_k^{(1)}) - f(p_{k-1}^{(1)}))/2 \leqslant d_{k-1}(p_k, p_{k+1}) \leqslant \delta^*, \end{aligned}$$

and

$$\begin{aligned} &\sup\{(f(y) - f(x))/2: p_{k-2} \leqslant x \leqslant p_{k-1}, p_{k-1} \leqslant y \leqslant p_{k-1}^{(1)}\} \\ &\quad \leqslant (M(p_{k-1}, p_{k-1}^{(1)}) - f(p_{k-1}))/2 \leqslant (M(p_{k-1}, p_{k-1}^{(1)}) - f(p_{k-1}^{(1)}))/2 \leqslant \delta^*, \end{aligned}$$

as shown above. Hence, $d_{k-1}(p_{k-2}, p_{k-1}^{(1)}) \leqslant \delta^*$. The proof is complete.

Define subsets Q and Q^* of Ω by

$$Q = \{ p \in \Omega: M(p_{i-1}, p_i) = M(p_i, p_{i+1}), \quad i \text{ odd, and}$$
$$m(p_{i-1}, p_i) = m(p_i, p_{i+1}), \quad i \text{ even, where } 1 \leqslant i \leqslant n-1 \},$$
$$Q^* = \{ p \in \Omega: f(p_i) = M(p_{i-1}, p_{i+1}), \quad i \text{ odd, and}$$
$$f(p_i) = m(p_{i-1}, p_{i+1}), \quad i \text{ even, where } 1 \leqslant i \leqslant n-1 \}.$$

Clearly, $Q^* \subset Q$. We call Q^* the set of alternant local extremal points of f. These sets play an important role in the analysis. The proof of the following proposition provides an iterative procedure for constructing a best knot vector in Q^* from a given initial best knot vector.

PROPOSITION 3.1. *Let $f \in C$. Then $P^* \cap Q^* \neq \varnothing$, and consequently, $Q \supset Q^* \neq \varnothing$.*

Proof. By Theorem 2.2 (b), $P^* = P \neq \varnothing$. Let $p \in P$. Also, let k be the smallest index such that $f(p_k)$ does not assume, on $[p_{k-1}, p_{k+1}]$, its local maximum for odd k or local minimum for even k, where $1 \leqslant k \leqslant n-1$. We first consider the case when k is odd. Find $p_k^{(1)} \in [p_{k-1}, p_{k+1}]$ such that $f(p_k^{(1)}) = M(p_{k-1}, p_{k+1})$ and replace p_k by $p_k^{(1)}$. If $k=1$ then let $p_0^{(1)} = p_0$. Now suppose that $k \geqslant 3$. If $f(p_{k-1}) = m(p_{k-2}, p_k^{(1)})$, then let $p_i^{(1)} = p_i$, $i = 0, 1, ..., k-1$. Otherwise we deduce $p_k < p_k^{(1)}$ and $m(p_k, p_k^{(1)}) < f(p_{k-1})$. Let

$$p_{k-1}^{(1)} = \inf\{z \in [p_k, p_k^{(1)}]: f(z) = m(p_k, p_k^{(1)})\},$$

and replace p_{k-1} by $p_{k-1}^{(1)}$. By Lemma 3.1, we have $d_{k-1}(p_{k-2}, p_{k-1}^{(1)}) \leqslant \delta^*$, $d_k(p_{k-1}^{(1)}, p_k^{(1)}) \leqslant \delta^*$, and $d_{k+1}(p_k^{(1)}, p_{k+1}) \leqslant \delta^*$. Also, we have $f(p_{k-1}^{(1)}) = m(p_{k-2}, p_k^{(1)})$ and $f(p_k^{(1)}) = M(p_{k-1}^{(1)}, p_{k+1})$. If $f(p_{k-2}) = M(p_{k-3}, p_{k-1}^{(1)})$, then let $p_i^{(1)} = p_i$, $i = 0, 1, ..., k-2$. Otherwise we deduce that $p_{k-1} < p_{k-1}^{(1)}$ and $M(p_{k-1}, p_{k-1}^{(1)}) > f(p_{k-2})$, and let

$$p_{k-2}^{(1)} = \inf\{z \in [p_{k-1}, p_{k-1}^{(1)}]: f(z) = M(p_{k-1}, p_{k-1}^{(1)})\}.$$

Replace p_{k-2} by $p_{k-2}^{(1)}$. Thus, $d_{k-2}(p_{k-3}, p_{k-2}^{(1)}) \leqslant \delta^*$, $d_{k-1}(p_{k-2}^{(1)}, p_{k-1}^{(1)}) \leqslant \delta^*$, $d_k(p_{k-1}^{(1)}, p_k^{(1)}) \leqslant \delta^*$, and $d_{k+1}(p_k^{(1)}, p_{k+1}) \leqslant \delta^*$, with $f(p_{k-2}^{(1)}) = M(p_{k-3}, p_{k-1}^{(1)})$, $f(p_{k-1}^{(1)}) = m(p_{k-2}^{(1)}, p_k^1)$, and $f(p_k^{(1)}) = M(p_{k-1}^{(1)}, p_{k+1})$.

By repeating this procedure, we obtain $p_{k-3}^{(1)}, ..., p_2^{(1)}, p_1^{(1)}$ such that

(a) $(p_0^{(1)}, p_1^{(1)}, ..., p_k^{(1)}, p_{k+1}, ..., p_n) \in P$, with $p_0^{(1)} = p_0$,

(b) $f(p_i^{(1)}) = M(p_{i-1}^{(1)}, p_{i+1}^{(1)})$, $i = 1, 3, ..., k-2$, $f(p_i^{(1)}) = m(p_{i-1}^{(1)}, p_{i+1}^{(1)})$, $i = 2, 4, ..., k-1$, and $f(p_k^{(1)}) = M(p_{k-1}^{(1)}, p_{k+1})$.

Let $p_i^{(1)} = p_i$, $i = k+1, ..., n$. Then, $p^{(1)} \in P$. We apply the same procedure to $p^{(1)}$ and obtain $p^{(2)}$. Continuing in this manner, in $r \leqslant n$ iterations we obtain a required $p^{(r)} \in P \cap Q^*$. If k is even, we use a similar construction to find an element in $P \cap Q^*$. The proof is complete.

The proofs of the following main results depend on the nonemptiness of the sets $P \cap Q^*$, Q^*, and Q as shown in the above proposition. We denote by $C^\infty = C^\infty(I)$ the set of all infinitely differentiable functions on I.

THEOREM 3.1 (Best approximation from $K(p) \cap C$). *Let $n \geqslant 2$, $p \in \Omega$ and $f \in C$.*

(a) *(Existence) Both $\underline{g}_p$, $\bar{g}_p \in K^*(p) \cap C$ if and only if $p \in Q$, and for such p, $\Delta(p) = \rho(f, K(p)) = \rho(f, K(p) \cap C)$.*

(b) *(Nonuniqueness) Suppose that $p \in Q^*$. Then $K^*(p) \cap C$ (and hence $K^*(p)$) is not a singleton if and only if $f \in C \backslash K(p)$.*

(c) *(C^∞ approximation) If $p \in Q$ and $f \in C \backslash K(p)$, then there exists $g \in K^*(p) \cap C^\infty$, and hence, $\Delta(p) = \rho(f, K(p)) = \rho(f, K(p) \cap C^\infty)$.*

Proof. (a) By Theorem 2.2(b), $\underline{g}_p$ and $\bar{g}_p$ are in $K^*(p)$. As in [8, part I], we conclude that both $\underline{g}_p(x)$ and $\bar{g}_p(x)$ are continuous at $x \neq p_i$ for $1 \leqslant i \leqslant n-1$, right-continuous at p_i for $0 \leqslant i \leqslant n-1$, and left-continuous at p_n. Suppose that $p \in Q$. Then, using the definition of Q, it is easy to verify that $\underline{g}_p(p_i^-) = M(p_{i-1}, p_i) - \delta(p) = M(p_i, p_{i+1}) - \delta(p) = \underline{g}_p(p_i)$, for i odd. Also, $\underline{g}_p(p_i^-) = f(p_i) - \delta(p) = \underline{g}_p(p_i)$, for i even. Hence, $\underline{g}_p \in C$. Similarly, $\bar{g}_p \in C$. Conversely, if $\underline{g}_p$ and $\bar{g}_p \in C$, then, using the definition of these functions and arguing as above at p_i, we conclude that $p \in Q$.

(b) By (a), if $K^*(p) \cap C$ is a singleton then $\underline{g}_p = \bar{g}_p$. Then, by the definition of Q^* we have $f(p_1) - \delta(p) = \underline{g}_p(p_1) = \bar{g}_p(p_1) = f(p_1) + \delta(p)$. This gives $\delta(p) = 0$ and $f \in K(p)$. The converse is obvious.

(c) This is established for $n = 1$ in [8, part I]. The proof for $n \geqslant 2$ is similar.

The proof is complete.

THEOREM 3.2 (Best Approximation from $K \cap C$). *Let $n \geqslant 2$, $f \in C$, and $g \in K \cap C$.*

(a) *(Existence) Both $\underline{g}_p$, $\bar{g}_p \in K^* \cap C$ if and only if $p \in P \cap Q$. Also, $\Delta^* = \rho(f, K) = \rho(f, K \cap C)$.*

(b) *(Characterization) $g \in K^* \cap C$ if and only if $\underline{g}_p \leqslant g \leqslant \bar{g}_p$, where $p \in P$.*

(c) *(Nonuniqueness) $K^* \cap C$ (and hence K^*) is not a singleton if and only if $f \in C \backslash K$.*

(d) *(C^∞ approximation) If $f \in C \backslash K$, then there exists $g \in K^* \cap C^\infty$. Hence, $\Delta^* = \rho(f, K) = \rho(f, K \cap C^\infty)$.*

Proof. (a) Clearly, g_p, $\bar{g}_p \in K^* \cap C$ if and only if g_p, $\bar{g}_p \in K^*(p) \cap C$, where $p \in P$. The result now follows from Theorem 3.1(a).

(b) As above, $g \in K^* \cap C$ if and only if $g \in K^*(p) \cap C$, where $p \in P$. The result then follows from Theorem 2.2(b).

(c) Suppose that $K^* \cap C$ is a singleton. Let $p \in P \cap Q^*$. Since $Q^* \subset Q$, by (a) g_p, $\bar{g}_p \in K^* \cap C$. Then g_p, $\bar{g}_p \in K^*(p) \cap C$. Since $K^* \cap C$ is a singleton, we have $g_p = \bar{g}_p$ and hence, $K^*(p) \cap C$ is a singleton. By Theorem 3.1(b), we conclude that $f \in K(p) \subset K$. The converse is obvious.

(d) This is established for $n = 2$ in [10] using methods of [8, part I]. The proof for $n \geqslant 2$ is similar.

The proof is complete.

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